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Bayesian Method for Quality Control with Weibull Distribution

Cecilia Novianti Salsinha[1]*, Yeni Rafita Sihombing[2], Melissa Aeudia Daullu[3]
[1]Department of Mathematics Education, Universitas Timor, Nusa Tenggara Timur, Indonesia
[2]Department of Management, Universitas Prima Indonesia, Sumatera Utara, Indonesia
[3]Department of Physics Engineering, Universitas International Papua, Papua, Indonesia
*Correspondence: ceciliasalsinha@unimor.ac.id

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ABSTRACT

Weibull distribution is a continuous probability distribution whose flexibility can be seen from the change in the shape of the distribution. If the scale and shape parameters are changed, then the Weibull Distribution will change into various distributions such as gamma and exponential distributions. Quality control, especially those that utilize reliability data, is one of the well-known applications of this distribution. The most common quality control used is the control chart. Because reliable data is often not normally distributed, Shewhart control graphs, which are commonly used, cannot be applied automatically. One way to solve this is by using the Bayesian method to form a control boundary. In applying the Bayes method, the prior is assumed to be uniformly distributed for the shape parameters Bayesian, and the reliability variable is assumed to be distributed inverse Weibull. The likelihood function is then formed by utilizing the reliability function of the Weibull distribution. The posterior distribution is obtained by combining the prior distribution and the likelihood function which is then used as a control limit by looking for the mean. The selection of LCL and UCL, [0.05, 195], produces PCR = 1.0. For normal Gaussian data this indicates that there are about 27 units that does not meet per 10,000 were produced. It can be concluded that the quality control process above is already good enough. However, this research is only limited to reliability data so it cannot be generalized if the data used is not reliability data.

Keywords: Weibull distribution, characteristics of Weibull distribution, reliability, quality control, Bayesian method

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Introduction

Development of science and technology positively impacts on various areas, including the field of Statistics. It can be seen from the growing number of inventions in the field of statistics, one of which is Distrubusi Weibull developed between 1922 and 1943 and named Weibull distribution based on the name of a machinist in Sweden, Waloddi Weibull (Lai, 2006).

Weibull distribution has been widely used by statistical experts in various fields of application of statistics. This is because the excess of this distribution can be used in various fields such as lifetime data analysis, weather data, and even other observations in economics, hydrology, and biology (Al-Zahrani et al., 2016). This distribution is known as a flexible distribution (Petropoulos et al., 2022).

Based on the density of the Weibull function, it can be seen that when $\beta = 1$, the density function is thus taken over will be a function of Exponential distribution with $\lambda = \alpha$. The exponential distribution is very useful, especially in analyzing reliability data. This is because the data reliability is generally not

Gaussian normal so it can't be solved using standard statistical methods which are commonly used for normal distribution. Some argue that the normal distribution is inappropriate for modeling lifetime data because the left-hand limit of the distribution extends to negative infinity pdf This could conceivably result in modeling negative times-to-failure. However, provided that the distribution in question has a relatively high mean and a relatively small standard deviation, the issue of negative failure times should not present itself as a problem (ROOHI, 2003). However one of the drawbacks of this distribution is only can be used if the component of failure rate is assumed to be constant, but in many cases, the failure rate component is not always constant (Limpert & Stahel, 2011). To address the problem of the failure of the components which can be a very long testing period, Weibull distribution are available (Luko, 1999)(Zhang et al., 2019).

The other application of the Weibull distribution is in the field of quality control. In 1924 Shewhart developed the control chart which is used as a tool in quality control. Control charts that are commonly used are the control charts that are based on a Normal distribution or 3σ control chart. However, in reality, not all the obtained data follows normal distribution. If the data is still being analyzed by the Shewhart control chart and it's assumed to be a normal distribution then it will be a fairly large error (Black et al., 2011)

To solve the problem of the many ways that is done, one of them is by using the quantile of Weibull distribution which is the inverse of the cumulative distribution function of the Weibull distribution. Another method used is the Bayesian method. This method is used to answer the problem of the scarcity of data because some data is primarily reliability data, the reliability is high enough so that it is difficult to find sufficient amounts of data (Lu et al., 2018). Therefore the use of the classic estimation methods (non-bayesian) became impossible, so in this research, the author tries to apply the Weibull distribution in the field of quality control by using the Bayesian method and the data used is reliable data (Guo, 2019).

Methods

Essentially the Bayesian approach to statistics is different from the classical approach in find estimator. In the classical approach, a parameter θ is something unknown then taken random samples $X_1, X_2, \ldots, X_n$ from the population with parameter θ and based on the values observed in the sample, knowledge of θ can be obtained, whereas in the Bayesian approach, the parameter θ is viewed as the quantity in which the variation is described by a probability distribution (known as the prior distribution). Then taken sample with the parameter θ and combined with the prior information on the parameters so that the posterior distribution of the parameters is obtained. The posterior distribution is obtained via Bayes rule so that the theorem is known as Bayesian statistics (Guo, 2019).

The following will be explained about the Bayes Theorem to understand Bayesian approaches. Suppose that $A_1, A_2, \ldots$ is the mutually exclusive event in the sample space Ω and let B be the event on Ω with $P(B) > 0$, then for any k

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} \tag{1}$$

((Sheldon et al., 1991)

Suppose that $X \sim f(x|\theta)$ and $\theta \in \Omega$. If the distribution of θ at Ω is being $\pi(\theta)$ then $\pi(\theta)$ is called the prior distribution of θ. The summary of the model can be expressed as

$$x|\theta \sim f(x|\theta) \tag{2}$$

$$\theta \sim \pi(\theta)$$

The following explanation will be discussed further on the 3 types of prior distribution namely:

1. Conjugate Prior

   Distribution Families of parametric $\mathcal{P}$ from distribution $\pi(\theta)$ is called a conjugate family for a family of $\mathcal{F}$ that corresponding distribution $\pi(x|\theta)$ if

$$\pi(\theta) \in \mathcal{P} \Rightarrow \pi(x|\theta) \in \mathcal{F} \tag{3}$$

   ((Nabilah et al., 2014))

2. Non-informative Prior Distribution

   If we don't know more about the parameter θ from the space of the parameters, then it can be taken the values of θ between $-\infty$ through ∞. At the condition without the information then non-informative prior can be used. Prior $\pi(\theta)$ is called non-informative prior if

$$\pi(\theta) \propto c \;\; for \; -\infty < x < \infty \tag{4}$$

   where c is any constant.(Sheldon et al., 1991)

3. Proper Prior

   Prior $\pi(\theta)$ is called proper prior if not depends on data and

$$\begin{cases} \int \pi(\theta) d\theta = 1, \theta \; continous \\ \sum_{\theta} \pi(\theta) = 1, \theta \; discrete \end{cases} \tag{5}$$

   A prior is not a proper prior if not satisfy the conditions above.

   (Sheldon et al., 1991)

Suppose that $X_1, X_2, \ldots X_n$ is a random sample of conditional distribution X given θ with probability density functions $f(x|\theta)$, then marginal probability density functions that is derived from $\underline{X} = (X_1, X_2, \ldots X_n)$ given θ is

$$f(\underline{x}|\theta) = f(x_1|\theta) f(x_2|\theta) \ldots f(x_n|\theta) \tag{6}$$

The joint probability density function of $\underline{X}$ and θ is

$$f(\underline{x},\theta) = f(\underline{x}|\theta)\pi(\theta) \tag{7}$$

so the conditional probability density function of θ given $\underline{X}$ that is

$$\pi(\theta|\underline{x}) = \frac{f(\underline{x},\theta)}{m(\underline{x})} = \frac{f(\underline{x}|\theta)\pi(\theta)}{m(\underline{x})} \tag{8}$$

$\pi(\theta|\underline{x})$ is called posterior distribution with $m(\underline{x})$ is **the** marginal probability density function of $\underline{X}$. Note that $f(\underline{x}|\theta)$ is the distribution of observed data $\underline{X}$ with $\underline{X}$ as a variable the value is known. When $\underline{X}$ is an observed value then the value of $m(\underline{x})$ is a constant or does not depend on θ and $f(\underline{x}|\theta)$ better known as the Likelihood function $L(\theta|\underline{x})$ *so*

$$\pi(\theta|\underline{x}) = c(\underline{x})L(\theta|\underline{x})\pi(\theta) \quad , c(\underline{x}) = \frac{1}{m(\underline{x})} \tag{9}$$

Thus, $\pi(\theta|\underline{x})$ is proportional with $L(\theta|\underline{x})\pi(\theta)$ and can be written as

$$\pi(\theta|\underline{x}) \propto L(\theta|\underline{x})\pi(\theta) \tag{10}$$

The form above is referred to the writing of Box-Tiao (Lai, 2006)

In many production processes, regardless of how well designed, a certain amount of natural variability will always exist. Natural variability is the cumulative effect of small reasons that are basically uncontrolled (hereafter referred to as the chance causes of variation). Within the framework of statistical control, a process that works only with a variety of unexpected reasons is said to still be in control, while the variability of sources that are not part of the pattern for the unexpected is called assignable causes of variation (Abdel-Motaleb, 2022). A process that works with any assignable causes of variation is called uncontrolled.

Here are some things related to Statistical process control:

1. Control chart

A control chart is the most popular tool used in quality control to control a process over and over. According to Shewhart's views cited by Alberto Luceno in the International Encyclopedia of Statistical Science, control charts are very useful in setting standards of attainment of a process, help to achieve these standards and consider which standards have been achieved (Şengöz, 2018). Examples of control charts can be seen in **Figure 1** below

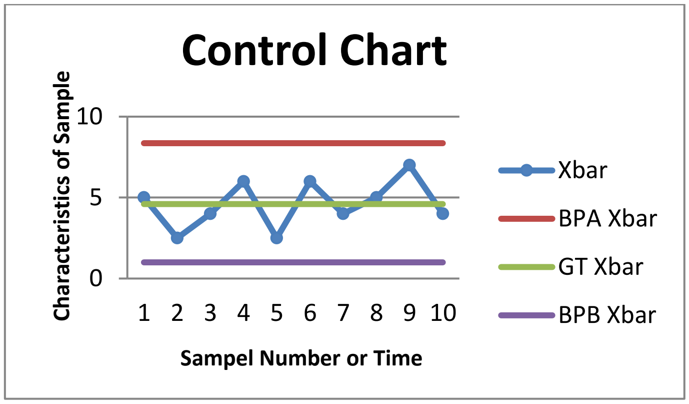

***Figure 1**. $\bar{X}$ Control Chart*

The control Chart at **Figure 1** is one of the Shewhart control chart. UCL (Upper Control Limit) and LCL (Lower Control Limit) are demonstrated by two horizontal lines which are red and purple on the chart. The chart also contains the CL (Central Limit) which is the average value of the quality characteristics that are associated with the under-control condition.

The control limit is selected such that when the process under control then the sample points will fall in between the two lines of it. If there is a point that is outside the limits then investigation and remedial action is required to remove the causes of such causes process so if omitted then the appearance of the process can be improved. When the process is under control then all points drawn should have essentially random patterns (Montgomery, 2013)

2. Process capability analysis

Process capability analysis can be defined as an environmental study in order to assess the ability of the process. Process capability assessments may be in the form of a probability distribution that has specifications of forms, the middle values (mean), and the spread (standard deviation). The most important is that the process capability analysis not only estimates the distribution of the characteristic quality of the product. Process capability analysis is a technique that has applications in many parts of the product, including product and process design, production planning, and production itself (Jeang, 2015).

A good method for explaining process capability is by calculating Process Capability Ratio (PCR)(Benkov et al., 2024) . Several methods have been introduced for calculating PCR, that is Clements method of percentile, Burr method of percentile, and by utilizing Cumulative Distribution Function (CDF s. Based on the simulated results of Ahmad et. al. by comparing the 3 methods mentioned above, obtained that the best method or method that approximates the best actual value is a method of CDF. Following is the translation of PCR with the calculation methods of CDF. Conventionally Process Capability Ratio is defined as:

$$PCR = \frac{BSA - BSB}{6\sigma} \tag{11}$$

If $X \sim N(\mu, \sigma^2)$ *then* $PCR = \frac{1}{3}\varphi_Z^{-1}\left(\frac{1}{2} + \frac{B}{2}\right)$ *and* $B = P(BSA < X < BSB)$.

With the cumulative Distribution Function method, the Process Capability Ratio is defined as follows:

$$PCR = \frac{\Phi^{-1}\left(0{,}5 + 0{,}5\int_{BSB}^{BSA} f(x)dx\right)}{3} \tag{12}$$

where f(x) indicates the density function of the process.

The research procedure can be shown in the diagram bellow.

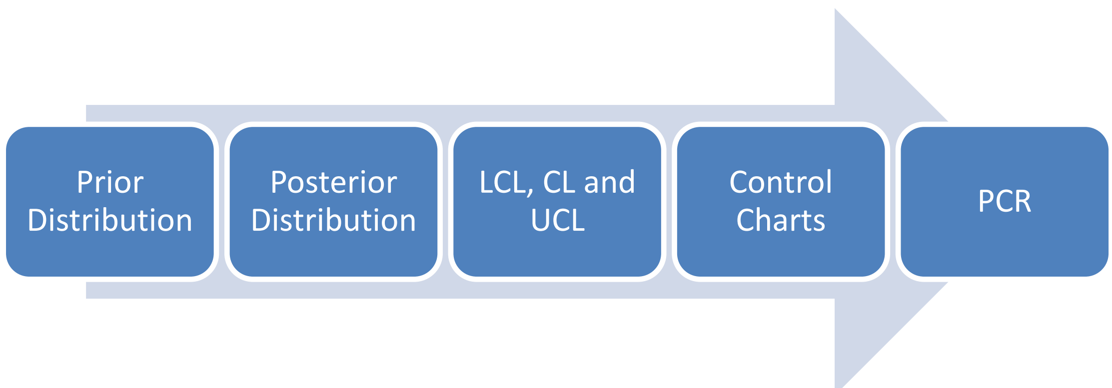

Figure 2. *The Research Procedure*

Results and Discussion

Together with the development of competition between companies in producing a product, the quality of the product is extremely important. A good product is achieved through good design and a lot of testing. One of these tests is the quality control method that utilizes the Bayesian method.

In the case of very small samples, the Shewhart control chart is not effective for the use (Erto & Giorgio, 2013). One of the ways used to solve this problem is to use estimator reliability based on the application of the Bayesian theorem which can be written as $P(A|B) = \frac{P(B|A)*P(A)}{P(B)}$,where $P(A)$ is the probability of A occurring, $P(B)$ is the probability of B occurring, $P(A|B)$ is the probability of B given A and $P(B|A)$ is the probability of A given B (Wu et al., 2021). This estimator is known as the Practical Bayes Estimator (PBE).

Note that the Bayesian approach to evaluation of reliability is not new for the Weibull distribution especially if the parameters are known. Many studies have discussed the use of the Bayes method in the analysis of reliable data based on Weibull distributions (Kumar & V, 2017)(Al-Bossly, 2020) (Guo, 2019)(Lu et al., 2018) (Zhang et al., 2019). If the inverse of the Gamma function is assumed to be the scale parameter and Uniform distribution for the shape parameter then the marginal posterior distribution can be found in closed form (Erto & Giorgio, 2013).

These some reasons why Weibull distribution is used in a model of reliability:

1. Characterized by two parameters (very rarely used third parameter); Although the estimation of Weibull distribution parameters with three parameters using traditional techniques such as maximum likelihood function is difficult because it is a nonlinear functions (Basheer & Algamal, 2021)
2. Flexibility of Weibull distribution which can be transformed into a variety of forms, depending on the parameter values (Ahmad & Hussain, 2017)
3. The meaning of the parameters is clear;
4. Shape parameter values are closely related to the nature of the failure mechanisms involved with;
5. Simple likelihood function is a function that is associated with the sample.

The reliability function of the Weibull distribution is as follows.

$$R(x,\alpha,\beta) = exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right) \quad ,x \geq 0; \quad \alpha,\beta > 0 \tag{13}$$

That can be reparameterized in x_R percentile and shape parameter so that the form of the reliability function would be:

$$R(x,x_R,\beta) = R^{\left(\frac{x}{x_R}\right)^{\beta}} = exp\left(-K\left(\frac{x}{x_R}\right)^{\beta}\right), x \geq 0;\ x_R,\beta > 0\ , K = l\,n(1/R) \tag{14}$$

Uniform prior density function on the interval (β_1, β_2) was assumed to correspond shape parameter β in sampling distribution:

$$f(x,\beta) = \begin{cases} \frac{1}{\beta_2-\beta_1};\ \beta_2 \geq \beta \geq \beta_1 > 0;\ \ \beta_2 > \beta_1 \\ 0 \qquad\qquad\qquad\qquad\qquad ;\ \ others \end{cases} \tag{15}$$

the equation above may be as infinite as it appears.

1. The formulation of the prior information on β

Prior Information of the β has been converted into the value β_1 and β_2 as specified in the equation above using the famous relationship between the failure mechanism and the value of β.

Both parameters can be effectively and easily anticipated, as already explained by Erto that information about the failure mechanism could always expressed in interval (β_1, β_2) (Erto & Giorgio, 2013). Therefore, this interval should be chosen large enough so that it actually contains a value of unknown parameters of the shape parameter of Weibull distribution.

To use the uniform distribution above as the prior distribution, it needs to limit of the parameters β_1 dan β_2 that is $\beta_1 + \beta_2 > 2$.

2. The assumption of the prior density function for the percentile x_R

The prior density function for x_R percentile selected (in accordance with the level of reliability that remains R) is assumed as the inverse Weibull (Erto & Giorgio, 2013):

$$f(x_R) = ab(ax_R)^{-(b+1)} \exp\left(-(ax_R)^{-b}\right), \quad x_R \geq 0,\ a, b > 0 \tag{16}$$

where a and b are scale parameter and shape parameter respectively. Please note that this distribution matched the prior information about x_R.

3. The assumption of equality of the prior and sampling shape parameters

Assumed the form $b = \beta$. In General, the larger β will cause the graphs of Weibull probability distribution functions will be increasingly peaked and the less uncertainty in x_R, so b must be greater. Therefore, $b = \beta$ is the simplest option. As a result, b cannot be determined in prior (17)

4. Expression of the prior belief about percentile x_R

The prior information for x_R must be converted only into the mean value $E(x_R)$ of the probability density function from equation (4) which is:

$$E(x_R) = \left(\frac{1}{a}\right) \Gamma\left(1 - \frac{1}{b}\right) \tag{17}$$

As the consequence of the assumption **(16)** and **(17)** above, the probability density function of x_R at (4) is converted into the conditional prior:

$$f(x_R|\beta) = a\beta(ax_R)^{-(\beta+1)} exp\left[-(ax_R)^{-\beta}\right]; \qquad x_R \geq 0;\ a, \beta > 0 \tag{18}$$

With anticipated mean $E\left(x_R\right)$.

5. Evaluation of the prior scale parameter a

After assumed $b = \beta$, parameter at the equation (19) is anticipated by reduced into a only. In addition, in order to simplify the procedure, in the equation (18) (and only in this equation) β can be replaced with the prior mean. Here is prior information about the prior for the scale parameter (a) and shape parameter (β)

$$a = \frac{\Gamma\left(1 - \frac{1}{\beta_m}\right)}{E\{x_R\}}; \qquad \beta_m = (\beta_1 + \beta_2)/2 \tag{19}$$

Usually in a reliability test, a set of random sample $\underline{x}$ from n experimental data is available so if reliability (measured in terms of lifetime) of the component is characterized into models on equations (15), then the Likelihood function of the sample is as follows.

$$L\left(\underline{x}|x_R, \beta\right) \propto \left(\frac{\beta}{x_R^{\beta}}\right)^n \prod_{i=1}^{n} x_i^{\beta-1}\, exp\left(-\frac{K}{x_R^{\beta}} \sum_{i=1}^{n} x_i^{\beta}\right) \tag{20}$$

By multiplying the prior at equations (3) and (6) then the joint prior density function of x_R and β are obtained as follows.

$$f(x_R,\beta) = (\beta_2-\beta_1)^{-1}a\beta(ax_R)^{-(\beta+1)}exp\left[-(ax_R)^{-\beta}\right] \quad (21)$$

Based on Bayes Theorem, then the joint posterior distribution of two parameters can be estimated as follows.

$$f(x_R,\beta|\underline{x}) = \frac{a^{-\beta}\beta^{n+1}x_R^{-\beta(n+1)-1}\prod_{i=1}^{n}x_i^{\beta-1}exp\left[-x_R{}^{-\beta}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right)\right]}{\int_{\beta_1}^{\beta_2}a^{-\beta}\beta^{n+1}\prod_{i=1}^{n}x_i^{\beta-1}\int_0^{\infty}x_R^{-\beta(n+1)-1}exp\left[-x_R{}^{-\beta}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right)\right]dx_Rd\beta} \quad (22)$$

The result above can be simplified so that we obtain

$$f(x_R,\beta|\underline{x}) = \frac{\beta^{n+1}a^{-\beta}x_R^{-\beta(n+1)-1}\prod_{i=1}^{n}x_i^{\beta-1}exp\left[-x_R{}^{-\beta}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right)\right]}{n!\int_{\beta_1}^{\beta_2}\beta^{n}a^{-\beta}\prod_{i=1}^{n}x_i^{\beta-1}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right)^{-(n+1)}d\beta} \quad (23)$$

Based on the posterior distribution above, we can estimate the parameter of x_R and β by find it's mean. $E(x_R|\underline{x})$ and $E(\beta|\underline{x})$ can be written simply as

$$E\{x_R|\underline{x}\} = \frac{I_3}{I_1}; \quad and \quad E\{\beta|\underline{x}\} = \frac{I_2}{I_1}, \quad (24)$$

Where;

$$I_j = \int_{\beta_1}^{\beta_2} a^{-\beta}\beta^{m_j}\prod_{i=1}^{n}x_i^{\beta-1}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right)^{-(n+1)+k_j}\Gamma(n+1-k_j)d\beta; \quad j=1,2,3$$

With the value of parameters m_j and k_j are as follows.

$$m_1 \equiv m_3 = n; \quad m_2 = n+1; \quad k_1 \equiv k_2 = 0; \quad k_3 = \frac{1}{\beta}$$

The very interesting thing that can be found is that the bivariate random variable (x_R,β) can be transformed into the standard Gamma distribution. The transformation is done using

$$y = x_R{}^{-\beta}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right) \quad (25)$$

To find the distribution of y will be used variable transformation with $\beta=\beta$. By utilizing the Jacobian transformation, we can find the joint probability density of y and β as follows.

$$f(y,\beta|\underline{x}) = \frac{\beta^{n}a^{-\beta}\prod_{i=1}^{n}x_i^{\beta-1}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right)^{-(n+1)}y^{n}exp[-y]}{n!\int_{\beta_1}^{\beta_2}\beta^{n}a^{-\beta}\prod_{i=1}^{n}x_i^{\beta-1}\left(a^{-\beta}+K\sum_{i=1}^{n}x_i^{\beta}\right)^{-(n+1)}d\beta} \quad (26)$$

So, obtained

$$f(y,\beta|\underline{x}) = \frac{y^{\gamma-1}exp[-y]}{\Gamma(\gamma)}; \quad y \geq 0; \; \gamma = n+1 \tag{27}$$

Based on the Shewhart control chart, the Central Line (CL) of the $x_R$$the$ control chart can be calculated utilizes all available data (all data $\underline{x}$) using the first estimator on equations (24). To find the estimation of $\hat{\beta}$ it can be used second estimator on equations (24). This estimation has inserted the inverse of the transformation on the equations (24) namely

$$x_R = y^{-\frac{1}{\hat{\beta}}}\left(a^{-\hat{\beta}} + K\sum_{i=1}^{n} x_i^{\hat{\beta}}\right)^{\frac{1}{\hat{\beta}}} \tag{28}$$

to obtain a good estimate of the percentile of $f(x_R, \beta|\underline{x})$ as a simple transformation from a percentile of the standard Gamma as shown above. This percentile is required as the control limit of the x_R control chart so that if given a false alarm risk α then Upper Control Limit (UCL) and Lower Control Limit (LCL) can be estimated namely $UCL = x_{R,\alpha/2}$ and $LCL = x_{R,1-\alpha/2}$ utilizes a simple transformation from a percentile $y_{1-\alpha/2}$ and $y_{\alpha/2}$ respectively from standard gamma distribution at the equation (26) ((Erto & Giorgio, 2013)). So, UCL, CL, and LCL can be written as

$$LCL = \left(y_{1-\alpha/2}\right)^{-\frac{1}{\hat{\beta}}}\left(a^{-\hat{\beta}} + K\sum_{i=1}^{n} x_i^{\hat{\beta}}\right)^{\frac{1}{\hat{\beta}}} \tag{29}$$

$$CL = \frac{\int_{\beta_1}^{\beta_2} \beta^{N+1} a^{-\beta} \prod_{i=1}^{N} x_i^{\beta-1} \left(a^{-\beta} + K\sum_{i=1}^{N} x_i^{\beta}\right)^{-(N+1)+\frac{1}{\beta}} \left(\frac{1}{\beta}\right) \Gamma\left(N+1-\frac{1}{\beta}\right) d\beta}{\int_{\beta_1}^{\beta_2} \beta^{N} a^{-\beta} \prod_{i=1}^{N} x_i^{\beta-1} \left(a^{-\beta} + K\sum_{i=1}^{N} x_i^{\beta}\right)^{-(N+1)} \Gamma(N+1) d\beta} \tag{30}$$

$$UCL = \left(y_{\alpha/2}\right)^{-\frac{1}{\hat{\beta}}}\left(a^{-\hat{\beta}} + K\sum_{i=1}^{n} x_i^{\hat{\beta}}\right)^{\frac{1}{\hat{\beta}}} \tag{31}$$

With false alarm risk $\alpha = 5\%$ and based on equation (27) $y \sim Gamma(1, n+1)$ so that $y_{1-\alpha/2} = y_{0.975}$ and $y_{\alpha/2} = y_{0.025}$.

In his research (Hsu et al., 2011) has found that Weibull control chart is more accurately than the percentile-Weibull control chart and Bootstrap-Weibull control chart. Using the adjusted process capability formula, the researchers determine the process capability of the Weibull control chart more accurately.

Table 1 contains the reliability of component data of ten samples with size $n = 5$ from one of the items in a company. That data is obtained from Prof. Fausto Galetto [Lecturer at Politecnico of Turin (teaching courses Quality Methods and Reliability Methods)] and has been divided by a factor of k which is determined by him. Data will be analyzed whether in control or out of control with a reliability rate of 99% and $x_R = 6{,}75$ hours. Data used in the Weibull analysis is presented in **Table 1** below.

Table 1. Product Reliability Data

Sample Number	Component 1	Component 2	Component 3	Component 4	Component 5
1	0,12	6,79	18,66	32,64	49,44
2	0,92	8,45	24,39	40,7	58,62
⋮	⋮	⋮	⋮	⋮	⋮
9	1,24	11,94	26,41	43,05	72,13
10	1,93	13,25	27,75	44,36	72,25

Before being analyzed, the data will be tested to know the data whether or not follow Weibull distribution. Based on the results above, can be seen that the resulting p-value was 0.53 which means that H_0 (the data follows Weibull distribution) is not rejected or it can be said that the data comes from Weibull distribution.

Data were analyzed by using the reliability rate $R = 0{,}99$ and $x_R = 6{,}75$. Based on previous data has been known that $n = 5$ dan $N = 50$. With the selection of the prior interval $[2{,}5\ ,\ 4{,}6]$ and assisted with the script R, here is a summary of the data obtained based on the above information.

Table 2. Summary of analysis results based on Bayesian method

No	Comp 1	Comp 2	Comp 3	Comp 4	Comp 5	xR	CL	LCL	UCL
1	0,12	6,79	18,66	32,64	49,44	5,612068	6,81144	5,151520	9,730307
2	0,92	8,45	24,39	40,7	58,62	6,73777	6,81144	5,151520	9,730307
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9	1,24	11,94	26,41	43,05	72,13	7,78105	6,81144	5,151520	9,730307
10	1,93	13,25	27,75	44,36	72,25	7,940219	6,81144	5,151520	9,730307

The data in Table 1 is then analyzed by utilizing equations (28), (29), and (30) to obtain CL, LCL, and UCL as shown in Table 2. Based on the result of UCL, CL, and LCL above then the control chart can be formed as follows

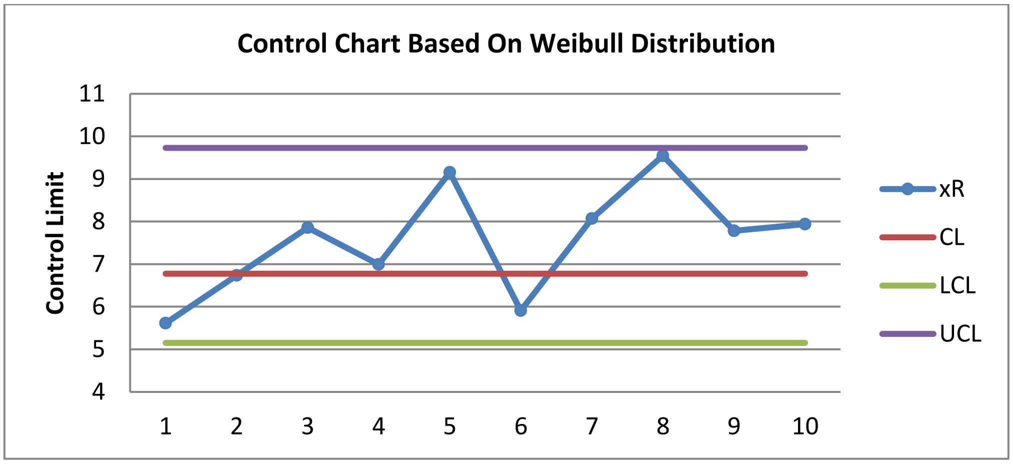

Figure 3. *Control chart based on the Bayesian method*

Based on the above, it can be seen that if the Weibull data is analyzed using a control chart using the Bayesian method, the result of the control chart can be said to be controlled because it is within the range of LCL and UCL. In addition, the distance between data and LCL or UCL is not as far as analyzed with a normal distribution. A control chart based on Normal distribution can be shown at **Figure 3**.

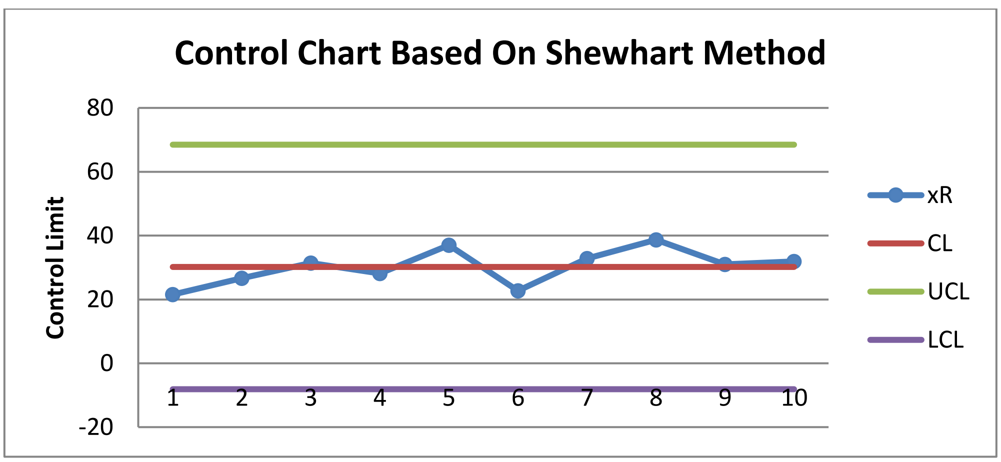

Figure 4. *Control Chart Based on the Shewhart Method*

Based on the above, it can be seen that if the Weibull data were analyzed by using the Shewhart control chart assuming normal data then it will be retrieved that the control chart will have a large range enough at the control limit that will allow all the data analyzed is concluded to be under control, despite the fact it is not. This is appropriate with the statement of (Aichouni & Bachioua, 2014) who said that if the data is not normal will be analyzed with a Shewhart control chart by assuming normal data then it will be a fairly large error occur. Now, in practice, as it is rather difficult to know the actual distribution, research about quality control finds a numerical comparison shows that our distribution- free EWMA TBEA chart performs as the parametric Shewhart TBEA chart, but without the need to pre-specify any distribution (Wu et al., 2021).

If the chosen value of UCL and LCL, namely [0.05, 195] then PCR calculations are as follows

$$\mathrm{PCR} = \frac{\Phi^{-1}(0{,}5 + 0{,}5 \cdot 0{,}99715)}{3} = 1{,}0$$

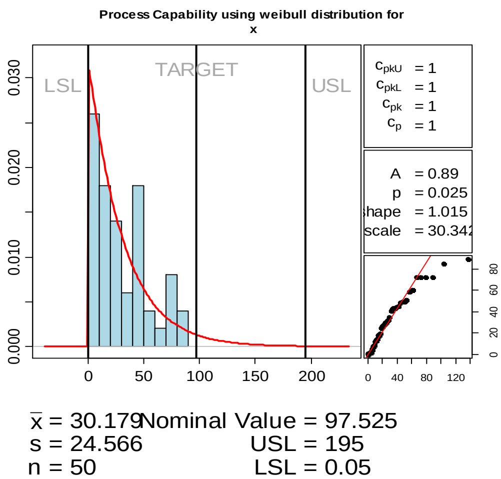

Figure 5. *Process capability ratio with R software*

Based on the above result, the selection of LCL and UCL, namely [0.05, 195] produces PCR = 1.0. For normal Gaussian data this indicates that there are about 27 units that does not meet 10,000 were produced. It can be concluded that the quality control process above is already good enough. But for the low process capability, companies need to carry out strict supervision of the performance of the production process in order to obtain the specifications desired by the company in meeting customer satisfaction (Madiana, 2022)

Conclusion

The Bayesian method and Weibull distribution can be used to build of a control-charts as an alternative of scarcity data and of course accommodate non-normal data. If non-normality data is analyzed using normal distribution will actually produce a considerable error. Based on the above result, the selection of LCL and UCL, namely [0.05, 195] produces PCR = 1.0. For normal Gaussian data, this indicates that there are about 27 units that does not meet per 10,000 were produced. It can be concluded that the quality control process above is already good enough. However, this research is only limited to reliable data so it cannot be generalized if the data used is not reliability data. Therefore, researchers can further analyze how the Bayesian method can be used in quality control for various data.

References

Abdel-Motaleb, H. (2022). Statistical Process Control. *Cutting Tool Engineering*, *74*(5), 32–35.

Ahmad, Z., & Hussain, Z. (2017). Flexible Weibull Extended Distribution. *MAYFEB Journal of Materials Science* , *2*(June 2017), 5–18.

Aichouni, M., & Bachioua, L. (2014). Control Charts for Non-Normal Data : Illustrative Example from the Construction Industry Business 2 Johnson ` s Distributions in Quality Improvement. *The 16th WSEAS International Conference on Mathematical and Computational Methods in Science and Engineering*, *2*, 71–76.

Al-Bossly, A. (2020). Bayesian statistics application on reliability prediction and analysis. *Journal of Statistics Applications and Probability*, *9*(1), 19–34. https://doi.org/10.18576/jsap/090103

Al-Zahrani, B., Marinho, P. R. D., Fattah, A. A., Ahmed, A. H. N., & Cordeiro, G. M. (2016). The (P-A-L) extended weibull distribution: A new generalization of the weibull distribution. *Hacettepe Journal of Mathematics and Statistics*, *45*(3), 851–868. https://doi.org/10.15672/HJMS.20156110040

Basheer, G. T., & Algamal, Z. Y. (2021). Reliability Estimation of Three Parameters Weibull Distribution based on Particle Swarm Optimization. *Pakistan Journal of Statistics and Operation Research*, *17*(1), 35–42. https://doi.org/10.18187/pjsor.v17i1.2354

Benkov, M., Bedn, D., & Bogdanovsk, G. (2024). *Process Capability Evaluation Using Capability Indices as a Part of Statistical Process Control*.

Black, G., Smith, J., & Wells, S. (2011). The impact of Weibull data and autocorrelation on the performance of the Shewhart and exponentially weighted moving average control charts. *International Journal of Industrial Engineering Computations*, *2*(3), 575–582. https://doi.org/10.5267/j.ijiec.2011.03.002

Erto, P., & Giorgio, M. (2013). *A note on using Bayes priors for Weibull distribution*. http://arxiv.org/abs/1310.7056

Guo, J. (2019). *An Introduction of Bayesian Method for Reliability Data Analysis Zhaojun (Steven) Li , Ph . D . & Jian Guo*. *January 2018*.

Hsu, Y. C., Pearn, W. L., & Lu, C. S. (2011). Capability measures for Weibull processes with mean shift based on Erto's-Weibull control chart. *International Journal of Physical Sciences*, *6*(19), 4533–4547. https://doi.org/10.5897/IJPS11.757

Jeang, A. (2015). Robust product design and process planning in using process capability analysis. *Journal of Intelligent Manufacturing*, *26*(3), 459–470. https://doi.org/10.1007/s10845-013-0802-6

Kumar, A. R., & V, K. (2017). A Study on System Reliability in Weibull Distribution. *Ijireeice*, *5*(3), 38–41. https://doi.org/10.17148/ijireeice.2017.5308

Lai, C. (2006). Springer Handbook of Engineering Statistics. *Springer Handbook of Engineering Statistics*, *May*. https://doi.org/10.1007/978-1-84628-288-1

Limpert, E., & Stahel, W. A. (2011). Problems with using the normal distribution - and ways to improve quality and efficiency of data analysis. *PLoS ONE*, *6*(7). https://doi.org/10.1371/journal.pone.0021403

Lu, D., Wang, L., Sun, Y., & Wang, X. (2018). Statistical Bayesian method for reliability evaluation based on ADT data. *IOP Conference Series: Materials Science and Engineering*, *351*(1). https://doi.org/10.1088/1757-899X/351/1/012008

Luko, S. N. (1999). A review of the weibull distribution and selected engineering applications. *SAE Technical Papers*, *September 1999*. https://doi.org/10.4271/1999-01-2859

Madiana, A. A. (2022). Process Capability Analysis for Quality Control at the Mixing Stage of Animal Feed Products at PT. Gold Coin Indonesia. *Mecomare*, *10*(4), 170–179. www.trigin.pelnus.ac.id

Montgomery, D.C. (2013). Introduction to Statistical Quality Control. Hoboken: John and Wiley

Nabilah, A. N., Soelistyo, A., & Kusuma, H. (2014). 済無No Title No Title No Title. *Paper Knowledge . Toward a Media History of Documents*, *5*(2), 40–51.

Petropoulos, F., Apiletti, D., Assimakopoulos, V., Babai, M. Z., Barrow, D. K., Ben Taieb, S., Bergmeir, C., Bessa, R. J., Bijak, J., Boylan, J. E., Browell, J., Carnevale, C., Castle, J. L., Cirillo, P., Clements, M. P., Cordeiro, C., Cyrino Oliveira, F. L., De Baets, S., Dokumentov, A., … Ziel, F. (2022). Forecasting: theory and practice. *International Journal of Forecasting*, *38*(3), 705–871. https://doi.org/10.1016/j.ijforecast.2021.11.001

ROOHI. (2003). *Some Problems in Realibility Theory.Ph.D.* 1–101.

Şengöz, N. G. (2018). Control Charts to Enhance Quality. *Quality Management Systems - a Selective Presentation of Case-Studies Showcasing Its Evolution*. https://doi.org/10.5772/intechopen.73237

Sheldon, N., Bain, L. J., & Engelhart, M. (1991). Introduction to Probability and Mathematical Statistics. *The Statistician*, *40*(4), 465. https://doi.org/10.2307/2348752

Wu, S., Castagliola, P., & Celano, G. (2021). A distribution-free EWMA control chart for monitoring time-between-events-and-amplitude data. *Journal of Applied Statistics*, *48*(3), 434–454. https://doi.org/10.1080/02664763.2020.1729347

Zhang, L., Jin, G., & You, Y. (2019). Reliability assessment for very few failure data and weibull distribution. *Mathematical Problems in Engineering*, *2019*. https://doi.org/10.1155/2019/8947905