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Extensions of Schauder's Fixed Point Theorem for Lipschitz Operators under Weakened Compactness Conditions

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ABSTRACT

This is a literature review focusing on the existence of fixed-point in Banach Space. This research aims to investigate the existence of fixed-point in Banach Space by examining the Schauder Fixed-Point Theorem and its extensions. This research began with a review of fundamental concepts such as metric space, compactness, convexity, and operator in Banach Space, the research proceeds to analyse the proof of the Schauder Fixed Point Theorem, which asserts that every continuous mapping on a compact and convex subset of Banach Space has a fixed point. This research further extends this result by examining cases where it is not the operator itself but the image of the operator that is compact yet still guarantees the existence of a fixed point. These findings broaden the applicability of Schauder Fixed-Point Theorem and suggest potential for its use in mathematical problems involving nonlinear-operators in infinite-dimensional spaces.

Keywords: Fixed point, Schauder, Lipschitz, Compact, Convex

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Introduction

Mathematics is a language and medium of communication of abstract things in the real world. Symbolic notation is the primary tool used in mathematics to express abstract ideas. The use of symbolic language has two advantages, namely simple and universal. Simple means concise, while universal implies that mathematicians worldwide can understand it. Before solving a mathematical problem, one must first determine whether a solution exists. For example, the Intermediate Value Theorem asserts that if a function f is continuous on the closed interval $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one $x \in [a, b]$ such that $f(x) = 0$, function f has at least one solution in the interval $[a, b]$ (Fernandez & Fernandez, 2022). The Intermediate Value Theorem is a classic example of an existence theorem. This theorem guarantees the existence of a solution without necessarily providing its explicit form.

Another significant theorem of existence solutions in mathematics is the fixed-point theorem. This fixed-point theorem has a very important role in functional analysis and operator theory. Fixed point theory is a branch of analysis with numerous applications (Pasangka, 2021). The application of this theory is to prove the existence of a solution to a system of differential equations.

There is Brouwer Fixed Point Theorem which states that continuous functions in closed, bounded and convex spaces have fixed points, which are in finite dimensional (closed ball in $\mathbb{R}^n$) (Ben-El-Mechaieh & Mechaiekh, 2022). While Schauder Fixed Point Theorem is in infinite dimensions (Hao et al., 2019). Brouwer Fixed Point Theorem is one of the most basic theorems in Topology, which states that every continuous function from a compact and convex set to itself has one fixed point. Schauder Fixed Point Theorem is an extension of Brouwer Fixed Point Theorem that applies to Banach spaces. Schauder Fixed Point Theorem states that a function that maps a compact and convex set to itself in a Banach Space has at least one fixed point.

However, the problem is that the existence and uniqueness of fixed-points become more complicated when in a Banach Space. In particular, the compactness condition required by Schauder's theorem may not always hold in practise. Therefore, this study aims to explore conditions under which the existence and uniqueness of fixed point can still be guaranteed in Banach spaces, focusing specifically on the use of Lipschitz operators.

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Methods

This research is classified as a literature study, by reviewing relevant literature on the concepts under study. The research examines sources or references that discuss the existence of fixed points in Banach space.

The stages of this research are as follows

1. Review of Relevant Literature

   This stage involves gathering relevant references including research articles and papers related to fixed-point theory and functional analysis.

2. Study of Supporting Concepts

   At this stage, the researcher reviews fundamental concepts that support the main result, including metric space, compactness, convexity and operators in Banach space.

3. Reconstruction of Proof

   The proof of the Schauder fixed point theorem was reconstructed based on authoritative sources. The logical steps and structure of the argument were systematically explained. This stage aimed to deepen theoretical comprehension rather than to introduce original proofs.

4. Analysis and Conclusion

   In this stage, the researcher develops logical arguments and applies the known conditions to prove the existence of fixed-point in Banach space, particularly for continuous and Lipschitz operators mapping into closed, bounded, and convex subsets. Finally, the researcher analyses the findings and draws conclusions, including the theoretical implications of the results and possible recommendations for further applications in mathematical problems involving nonlinear operators in infinite-dimensional spaces.

Through this approach, the study will contribute a deeper conceptual understanding of fixed-point existence within the framework of the Schauder theorem and to enrich the body of literature in the field of functional analysis.

Results and Discussion

This term will present some basic definitions and important concepts as a theoretical basis in discussing the existence of Schauder Fixed-Point Theorem. The following definitions are needed to form an understanding of Metric space and the compactness properties which related to the fixed point theorem.

Definition 1 (Alzate, C. P. P., Sanchez, T. F., & Munoz, 2023)

Let X be a non-empty set. A Metric space or distance on X is a function $d: X \times X \to \mathbb{R}$ that satisfies the following properties:

a. $d(x, y) \geq 0$ for all $x, y \in X$ (*Non-negativity*)
b. $d(x, y) = 0$ if and only if $x = y$ (*Identity of Indiscernible*)
c. $d(x, y) = d(y, x)$ for all $x, y \in X$ (*Symmetry*)
d. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (*Triangle Inequality*)

A metric space (X, d) is a pair of an empty set X and a metric d on X.

Definition 2 (Kusumaningati & Jakfar, 2023)

A Subset F of Metric space (X, d) is called *G-sequentially compact* if whenever $x = \{x_n\}$ is sequence of points in F there is a subsequence $y = \{x_{n_k}\}$ of x with $G(y) \in F$.

After discussing the definition of Metric space and the compactness properties of sequence, the next term will present theorems and advanced concepts of Metric space and the concept of convexity in linear space.

Theorem 1 (Mykhaylyuk & Myronyk, 2020)

Let a Metric space (X, d) and $K \subseteq X$. Then the following three statements are equivalent

a. K is a complete and totally bounded set

b. K is compact set

c. K is a sequentially compact set.

Definition 3 (Abdulkarim, 2025)
A linear operator $T: X \to Y$ is said to be compact if it maps every bounded set of X into a relatively compact subset of Y, i.e., the closure of $T(B)$ is compact in Y for any bounded set $B \subseteq X$.

The concept of convexity plays a central role. Convex sets and convex function provide a foundational structure that supports many important results, including Schauder and Brouwer Fixed-Point Theorem.

Definition 4 (Kashuri et al., 2022)
A subset of the linear space X is said to be convex if for every distinct pair x and y of subset, it contains $\lambda x + (1-\lambda)y, \forall \lambda \in [0,1]$.

Definition 5 (Kashuri et al., 2022)
A function f is said to be convex on X if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

Holds for all $x, y \in X$ and for every $\lambda \in [0,1]$.

Definition 6 (Taipi, 2023)
Let A subset of the linear space. The Convex Hull of the set A is the intersection of all convex subsets of X containing A.

Corollary 1 (Taipi, 2023)
Let A subset of the linear space. The convex hull is

$$co(A) = \left\{\sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N}, a_1, \cdots, a_n \in A, \lambda_1, \cdots, \lambda_n \in \mathbb{R}^+ with \sum_{i=1}^{n} \lambda_i = 1\right\}.$$

Definition 7 (Melo, 2017)

Let X, Y be metric space, $f: X \to Y$ a continuous function at $p \in X$ and $\varepsilon > 0$. A positive number δ is said to be a delta-epsilon number for f at p, if δ atisfies the $\varepsilon - \delta$ definition of continuity of f at the point p. In other words, δ is such that

$$if\ x \in X\ and\ d_X(x,p) < \delta\ then\ d_Y\big(f(x), f(p)\big) < \varepsilon.$$

After understanding the foundational concepts of metric space, sequential compactness and convexity, next term will explore Schauder Fixed-Point Theorem and its generalizations.

Theorem 2

If X Banach Space and $K \subset X$ be compact and convex set then K has Fixed Point.

Proof

Given any continuous mapping $T: K \to K$ Since K is compact, so that K is bounded (Isabu & Ojiema, 2024). For every integer $\varepsilon > 0$ there is a finite set $\{x_1, x_2, \cdots, x_N\} \subset K$ such that for every $x \in K$ there is a $x_j, j = 1,2,\cdots, N$ such that $\|x - x_j\| < \varepsilon$.

Define $\{x_j\} \subset K$ for $j = 1,2,\cdots, N$,,

$$g_j(x) = \begin{cases} \varepsilon - \|x - x_j\| & , \|x - x_j\| < \varepsilon \\ 0 & , \|x - x_j\| \geq \varepsilon \end{cases}$$

We will prove that g_j is continuous in x for $j = 1,2,\cdots, N$.

For each number $\delta > 0$, the number η will be searched for such that for each $y \in K$ with $\|x - y\| < \eta$, $\|g_j(x) - g_j(y)\| < \delta$ applies. Obtained

$$\begin{aligned} \|g_j(x) - g_j(y)\| &= \|\varepsilon - \|x - x_j\| - (\varepsilon - \|y - y_j\|)\| \\ &= \big\|\|y - y_j\| - \|x - x_j\|\big\| \\ &\leq \|y - y_j\| + \|x - x_j\| < \delta \end{aligned}$$

Provided $\|x - y\| = \|x - x_j + x_j - y\| \leq \|x - x_j\| + \|y - x_j\| < \eta = \delta$.

It is proven that g_j is continuous on $x \in K$.

Formed

$$h_j(x) = \frac{g_j(x)}{\sum_{i=1}^{N} g_i(x)}$$

for each $x \epsilon K$

It will be shown that

$$\sum_{i=1}^{N} g_i(x) > 0$$

for each $x \epsilon K$

As follows

$$g_1(x) = \begin{cases} \varepsilon - \|x - x_1\| & , \|x - x_1\| < \varepsilon \\ 0 & , \|x - x_1\| \geq \varepsilon \end{cases}$$
$$g_2(x) = \begin{cases} \varepsilon - \|x - x_2\| & , \|x - x_2\| < \varepsilon \\ 0 & , \|x - x_2\| \geq \varepsilon \end{cases}$$
$$\vdots$$
$$g_2(x) = \begin{cases} \varepsilon - \|x - x_N\| & , \|x - x_N\| < \varepsilon \\ 0 & , \|x - x_N\| \geq \varepsilon \end{cases}$$

$$\sum_{i=1}^{N} g_i(x) = g_1(x) + g_2(x) + \cdots + g_N(x)$$

For any $x \in K$ there is $1 \leq j \leq N$ with $g_j(x) > 0$.

So $\sum_{i=1}^{N} g_i(x) > 0$ for every $x \in K$.

As a result h_j is continuous, because g_j is continuous and $\sum_{j=1}^{N} g_j(x) > 0$ for every $x \in K$, and $h_j(x) \geq 0$ and

$$\sum_{j=1}^{N} h_j(x) = \frac{\sum_{j=1}^{N} g_j(x)}{\sum_{j=1}^{N} g_j(x)} = 1$$

for every $x \in K$

Furthermore $h_j(x) = 0$ if $\|x - x_j\| \geq \varepsilon$.

For every $x \in K$ we form

$$V(x) = \sum_{j=1}^{N} h_j(x)\, x_j$$

Which defines a continuous mapping $: K \to K_0$, with K_0 being *Convex Hull* of $\{x_1, x_2, \cdots, x_N\}$ in other words

$$V(x) = \sum_{j=1}^{N} h_j(x)\, x_j$$

with $x_j \in K; 0 \leq h_j \leq 1;\ \sum_{j=1}^{N} h_j(x) = 1$

So for every $x \in K$,

$$x - V(x) = 1x - V(x) = \sum_{j=1}^{N} h_j(x)\, x - \sum_{j=1}^{N} h_j(x)\, x_j = \sum_{j=1}^{N} h_j(x) \left(x - x_j\right)$$

As a result

$$\|x - V(x)\| = \left\| \sum_{j=1}^{N} h_j(x)\left(x - x_j\right) \right\| = \|x - x_j\| < \varepsilon$$

for each $x \in K$. Since $T: K \to K$ and $V: K \to K_0$ are continuous, then $V \circ T: K \to K_0$ is continuous.

The function $V \circ T$ is in K_0 in finite-dimensional space. Based on Brouwer's Fixed Point theorem (Kamalov & Leung, 2023), there is a fixed point for $V \circ T$, called $x_\varepsilon \in K$, meaning

$$V\big(T(x_\varepsilon)\big) = x_\varepsilon$$

For $n = 1,2,3,\cdots,$, take $\varepsilon = \frac{1}{n}$. There is $x_n \in K$ such that $V_n\big(T(x_n)\big) = x_n$ for every n.

Consequently

$$\|V_n(y) - y\| < \frac{1}{n}$$

for every n.

Since the sequence $\{x_n\} \subset K, K$ is a Compact set, it has a subsequence $\{x_{n_k}\}$ (Yaying, 2020) that converges to a point $x^* \in K$, which can be written as $x_{n_k} \to x^*$

Since T is continuous, $T\big(x_{n_k}\big) \to T(x^*)$.

Obtained

$$\begin{aligned}\|T(x^*) - x^*\| &= \left\|T(x^*) - T\big(x_{n_k}\big) + T\big(x_{n_k}\big) - x_{n_k} + x_{n_k} - x^*\right\| \\ &\leq \left\|T(x^*) - T\big(x_{n_k}\big)\right\| + \left\|T\big(x_{n_k}\big) - x_{n_k}\right\| + \left\|x_{n_k} - x^*\right\| < \frac{1}{n}\end{aligned}$$

so that

$$T(x^*) = x^*$$

Thus, the continuous mapping has fixed points, the set K has fixed points.

Theorem 3 (Figueroa & Infante, 2016)

Let K be a nonempty, closed, bounded, convex subset of Banach space X and suppose that $T: K \to K$ is a compact operator that is T is continuous and maps bounded sets into *precompact* one. Then T has a fixed point.

Corollary 2 (Hao et al., 2019)

Let X be a Banach space and $C \subseteq X$ be a closed, convex and bounded set. If $T: C \to C$ is a Contraction operator and continuous, $T(C)$ is compact in C then there exists $x^* \in C$ such that

$$T(x^*) = x^*.$$

Corollary 3

Let X be a Banach space and $C \subseteq X$ be a closed, convex and bounded set (Kaya, 2018). If $T: C \to C$ is a Lipschitz operator and continuous, $T(C)$ is compact in C then there exists $x^* \in C$ such that

$$T(x^*) = x^*$$

Proof

Given X is a Banach space. The set $C \subseteq X$ is closed, convex and bounded. Assume that If $T: C \to C$ is continuous, in other words

$$|x - y| < \delta \Longrightarrow |T(x) - T(y)| < \varepsilon$$

In addition, T is a Contraction operator (Laha & Saha, 2016), in other words

$$|T(x) - T(y)| \leq L|x - y| \; with \; Lipschitz \; L \leq 1$$

Since C is compact then there exists a subsequence $\{x_{n_k}\}$ that converges in C (Yaying, 2020). This implies that $T(C) \subseteq C$ is also compact (Nakasho, K., Narita, K.,& Shidama, 2016).
Define

$$K \coloneqq \overline{conv}(T(C).$$

Where $conv\ (T(C))$ is the smallest closed convex set (Wu et al., 2019) and $K \subseteq T(C)$, $T(C) \subseteq C$ and C is closed and convex. Since $T(C)$ is compact then the convex $conv\ (T(C))$ is also compact. As a result, Schauder's Fixed Point Theorem (Figueroa & Infante, 2016) becomes that there is a fixed point.

The proof of corollary 3 establishes that a continuous and Lipschitz operator mapping into a closed, convex and bounded subset of a Banach Space and whose image is compact, necessarily possesses a fixed-point. This result, corollary 3 represents a refinement and extension of the classical Schauder Fixe-Point Theorem. While the original Schauder Fixed Point Theorem requires the operator itself to be compact, this corollary relaxes the condition by instead requiring the image of the operator to be compact. Consequently, it broadens the applicability of fixed-point theory to setting where verifying the compactness of the operator is difficult, but the compactness of its image can be ensured.

Conclusion

This research investigates the existence of fixed point in Banach spaces. The Brouwer Fixed-Point Theorem, a classical theorem, states that every continuous mapping on a compact and convex subset of a finite-dimensional space has a fixed point. This foundational theorem is extended by the Schauder Fixed

Point Theorem, which guarantees the existence of a fixed point for a continuous mapping on a nonempty, compact, and convex subset of Banach space.

Furthermore, the findings of this research demonstrate that the existence of fixed point is not restricted to compact operators. Even when the operator itself is not compact, the compactness of its image is sufficient to unsure the existence of a fixed-point. This insight significantly expands the scope of fixed-point theory.

Based on these results, the researcher recommends the application of Schauder-type fixed point theorems in addressing various mathematical problems involving nonlinear operators in infinite-dimensional spaces, especially where full compactness may not be attainable but image compactness can be established.

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